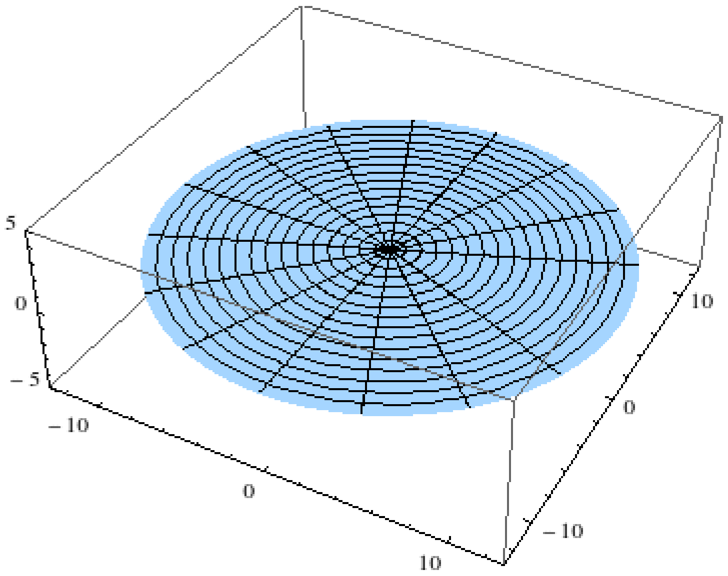
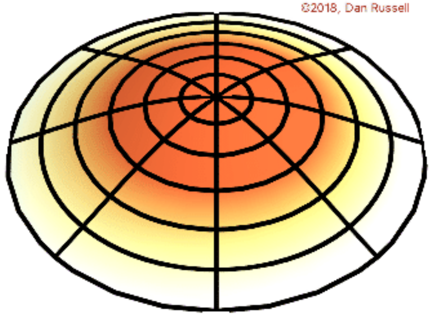


Circular membrane

2ND NOVEMBER 2020

https://www.acs.psu.edu/drussell/Demos/MembraneCircle/Circle.html



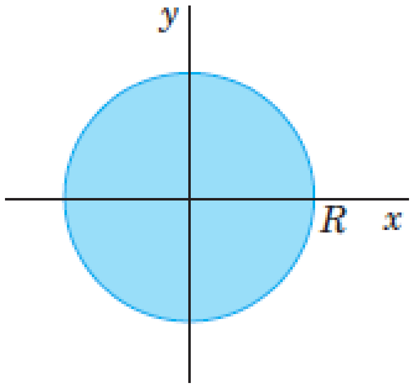
Laplacian polar coordinates

- Since we want to discuss the vibration of circular membranes, the most suitable Laplacian in the wave equation has to be described in Polar coordinates.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\ \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

- We only consider a membrane of radius R and determine the solution ϕ(r,t) that are radially symmetric.
- The wave equation becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\ \phi}{\partial r} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$



- And the boundary condition stating that ϕ(R,t) = 0 at all times

Solution of the wave equation : Bessel's equation

- Using the method of separation of variables, we first determine solution $\phi(r,t) = R(r)T(t)$.
- Substituting ϕ and its derivatives back into the wave equation and arrange the expression as follows

$$\frac{\ddot{T}(t)}{c^2 T(t)} = \frac{1}{R(r)}\left(R''(r) + \frac{1}{r} R'(r) \right) = -k^2$$

- Note that dots denote derivatives with respect to t and primes denote derivatives with respect to r.
- The expressions on both sides must equal a constant. This gives two linear ODEs,

$$\ddot{T}(t) + \lambda^2 T(t) = 0 \quad \text{where} \quad \lambda = ck \qquad (1)$$

$$R''(r) + \frac{1}{r} R'(r) + k^2 R(r) = 0 \qquad (2)$$

General form of Bessel's equation

- This is a general form of Bessel's equation

$$\frac{d^2}{dx^2}y+\frac{1}{x}\frac{d}{dx}y+\left(1-\frac{\nu^2}{x^2}\right)y=0$$

- Complete solution of the Bessel's equation is given as

$$y=CJ_\nu(x)+DY_\nu(x)$$

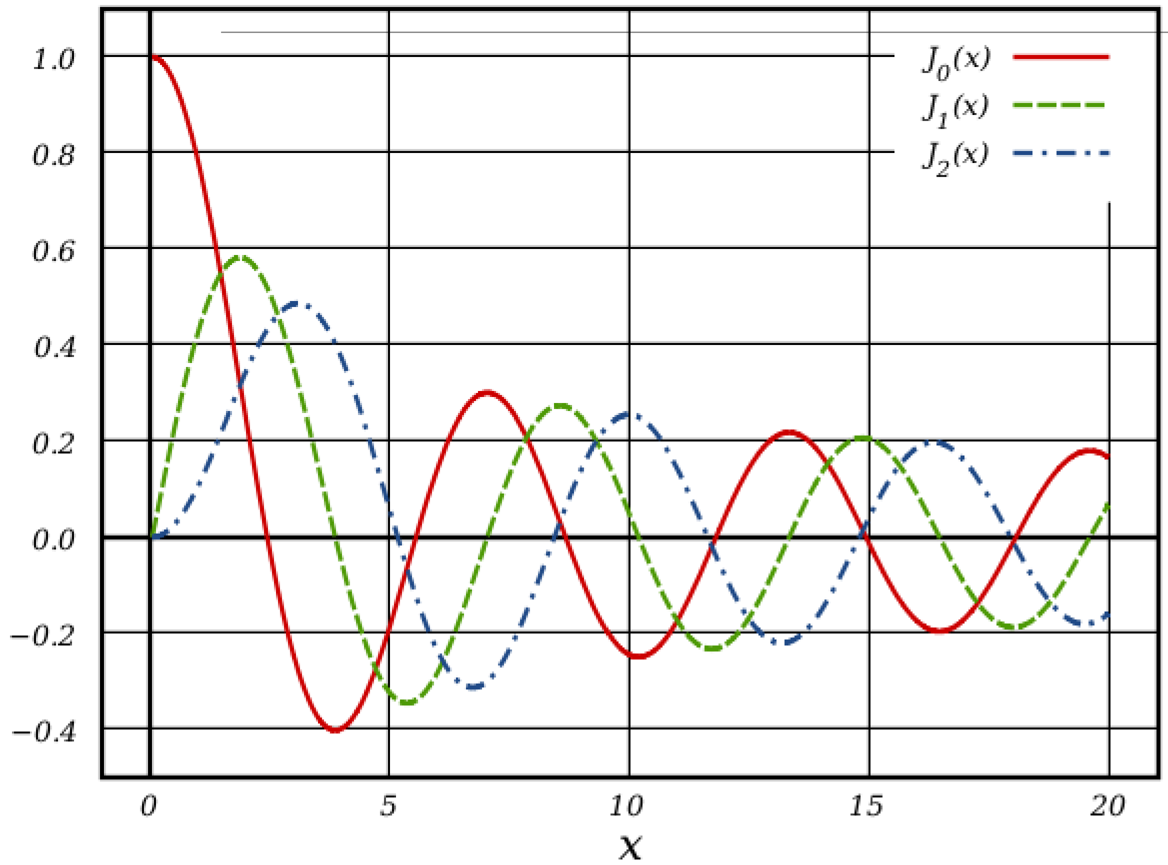
- Where ν is the order of the equation

  J_ν is the Bessel's function of the 1^{st} kind order ν

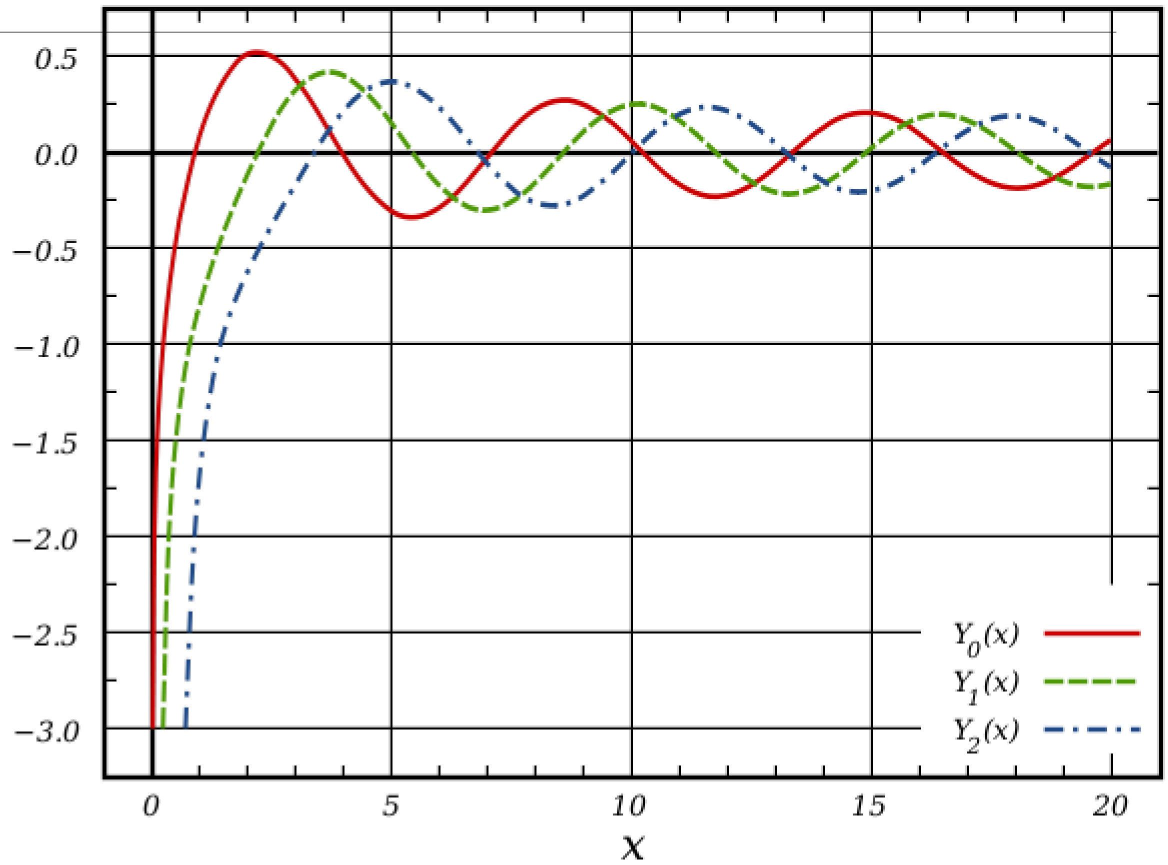
  Y_ν is the Bessel's function of the 2^{nd} kind order ν

  and C and D are arbitrary constants.

Bessel functions



Bessel's function of the 1st kind



Bessel's function of the 2nd kind

Solution of the linear ODE

- Recall the linear ODE eq. (2)

$$R''(r)+\frac{1}{r}R'(r)+k^2R(r)=0 \qquad (2)$$

- We can reduce (2) to Bessel's equation if we set s = kr. Then 1/r = k/s and, keeping the notation R for simplicity.

$$R'=\frac{dR}{dr}=\frac{dR}{ds}\frac{ds}{dr}=\frac{dR}{ds}k \quad \text{and} \quad R''=\frac{d^2R}{ds^2}k^2$$

- By substituting this into eq. (2) and omitting the common factor k^2 we have

$$\frac{d^2R}{ds^2}+\frac{1}{s}\frac{d\ R}{ds}+R=0 \qquad (3)$$

$$\frac{d^2R}{ds^2}+\frac{1}{s}\frac{d\ R}{ds}+R=0 \qquad (3)$$

$$\frac{d^2}{dx^2}y+\frac{1}{x}\frac{d}{dx}y+\left(1-\frac{\nu^2}{x^2}\right)y=0$$

Bessel's equation

- This is clear that eq.(3) corresponds to the expression of the Bessel's equation with order ν = 0.

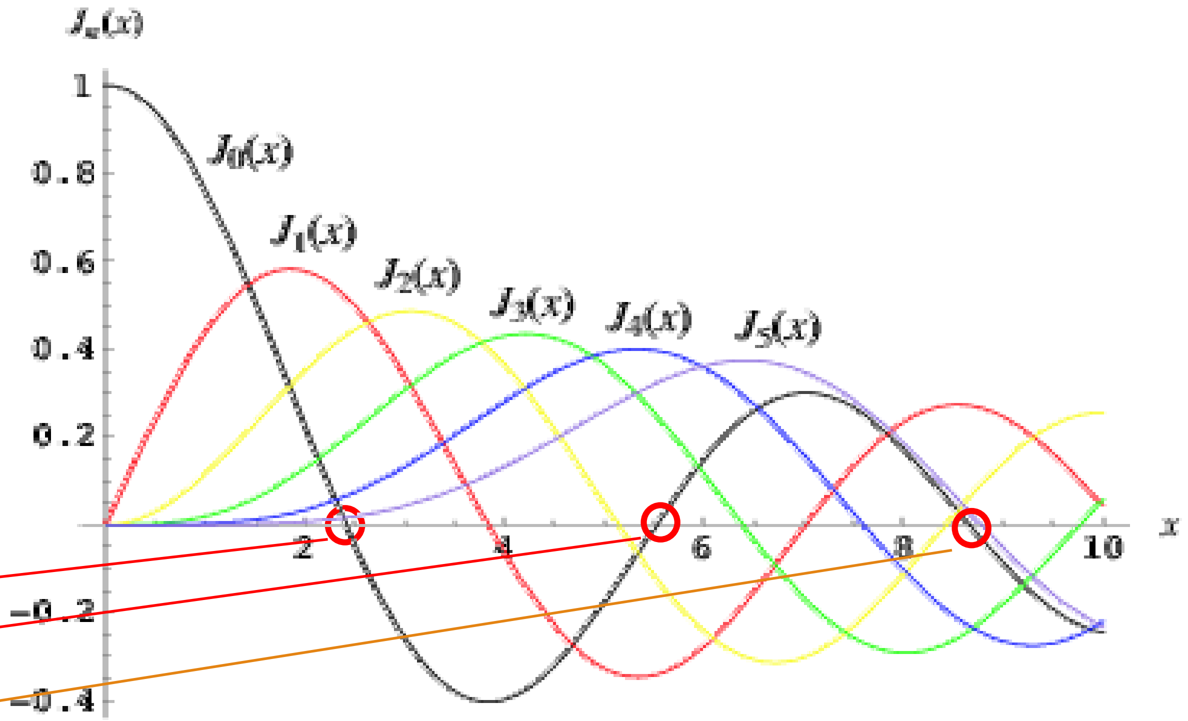
- Solutions of eq. (3) are Bessel function J_0 and Y_0 of the first and second kind.
- But **Y_0 becomes infinite at 0**, so that we cannot use it because the vibration of the membrane must always remain finite.
- Thus, the solution becomes

$$R=CJ_0(s)=CJ_0(kr) \qquad (4)$$

- Consider eq. (4) along with the boundary condition $\phi(R,t) = 0$

$$R = CJ_0\left(s\right) = CJ_0\left(kr\right) \qquad (4)$$

- We get R that satisfies this condition because J_0 has (infinitely many) positive zeros (shown in the Bessel function graph) , $s = \alpha_1, \alpha_2, \ldots$ with numerical values



$$\alpha_1 = 2.4048, \quad \alpha_2 = 5.5201, \quad \alpha_3 = 8.6537, \ldots$$

Bessel's function of the 1st kind

- Due to multiple values of argument that give $J_0 = 0$, a modification can be made as follows

$$kR = \alpha_m \quad \text{thus} \quad k = k_m = \frac{\alpha_m}{R}, \quad m = 1, 2, \ldots$$

- Hence, the functions $$R_m(r) = CJ_0(k_m r) = CJ_0\left(\frac{\alpha_m}{R} r\right) \qquad (5)$$

are solutions of eq. (2) that are zero on the boundary circle r = R.

- This suggests that the solution for linear ODE eq. (1) can be given as

$$T_m(t) = A_m \cos \lambda_m t + B_m \sin \lambda_m t \qquad (6)$$

- Where $\lambda = \lambda_m = ck_m = c\alpha_m / R$

A complete solution for wave function of a circular membrane

- Therefore, the **wave function with a radial symmetry** for a circular membrane can be written as $\phi(r,t) = R(r)T(t)$.

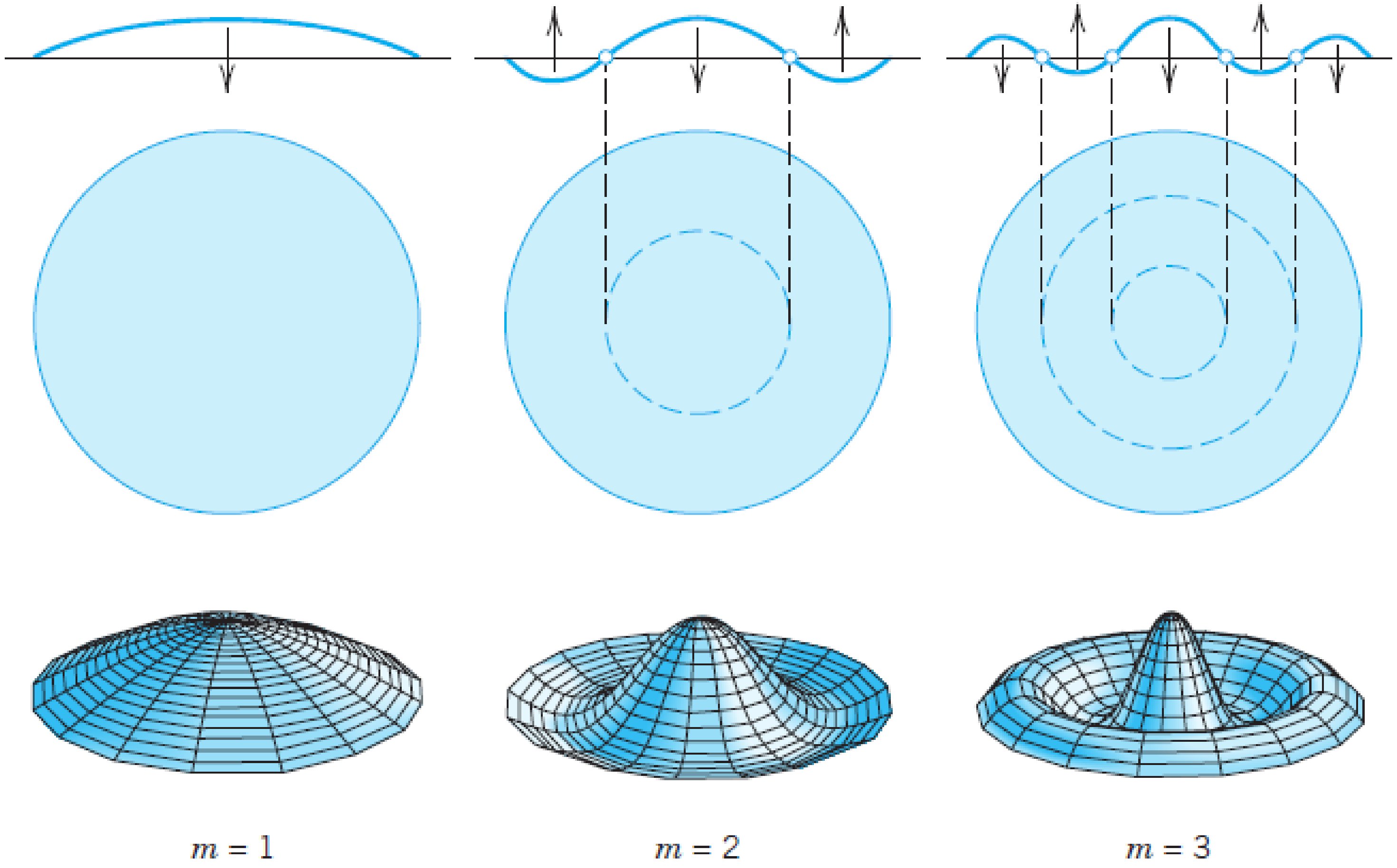
$$\phi(r,t) = R_m(r)T_m(t) = T_m(t) = (A_m \cos \lambda_m t + B_m \sin \lambda_m t) C J_0\left(\frac{\alpha_m}{R} r\right)$$

- The vibration pattern of the circular membrane is described by the radial function R.

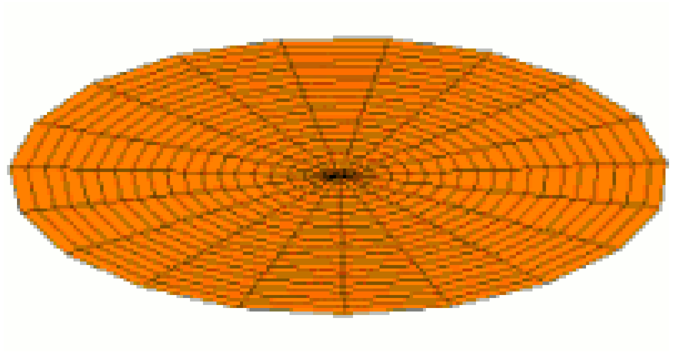
- For example : m = 1, and $0 < r < R$, ϕ becomes a maximum value when r = 0 and the function gradually reduced to zero; i.e. $J_0(\alpha_1) = 0$ at r = R,

m = 2 and $0 < r < R$, ϕ becomes a maximum value when r = 0 and the function becomes zeros twice; i.e. $J_0(\alpha_1) = 0$ and $J_0(\alpha_2) = 0$ at r = R. Under this condition, there is **one nodal line**.

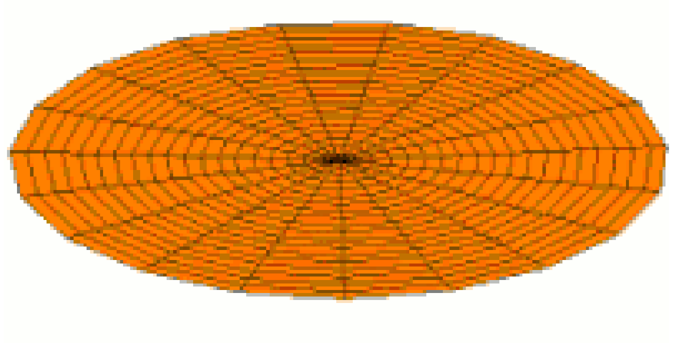
Normal modes of the circular membrane in the case of vibration independent of angle



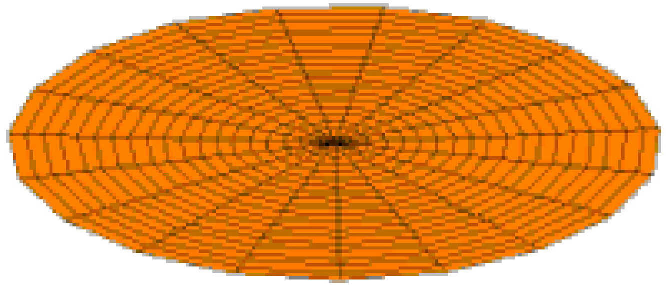
Vibration patterns of a circular membrane



m =1



m =2



m =3